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Magnetohydrodynamic flow due to the discharge of an electric current in a hemispherical container

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In this paper we consider the flow field induced in an incompressible viscous conducting fluid in a hemispherical bowl by a symmetric discharge of electric current from a point source at the centre of the plane end of the hemisphere. This plane end is a free surface. We construct an analytic solution for the slow viscous flow and a numeriacl solution for the nonlinear problem. The streamlines in an axial cross-section form two sets of closed loops, one on either side of the axis. Our computations indicate that, for a given fluid, when the discharged current reaches a certain magnitude the velocity field breaks down. This breakdown probably originates at the vertex of the hemispherical container.

1. Introduction

In some processes in arc welding and electrochemistry an electric current is passed through a conducting fluid. The current usually enters the fluid through a small area and diverges into the fluid. The Lorentz force due to this current and the associated magnetic field is rotational and sets the fluid in motion. In an attempt to gain insight into the structure of the flow field for this problem several authors have considered the case of an electric current discharged radially from a point source on a plane interface into a fluid extending to infinity. Thus Lundquist (1969) considered the linear problem (slow viscous flow), whereas Shercliff (1970) considered the nonlinear inviscid problem. Sozou (1971) considered the nonlinear viscous problem and Sozou & English (1972) investigated the case where there is interaction between the velocity and the electromagnetic field. Sozou & Pickering (1975) considered the development of the flow field in the nonlinear viscous problem. From all these studies it turns out that the flow field has a jet-like structure, similar to that of the momentum jet emerging from a hole of a plane bounding a semi-infinite fluid (Squire 1952). For a given fluid the nonlinear viscous velocity field breaks down when the discharged current exceeds a certain magnitude.

The above studies refer to the case where the fluid extends to infinity whereas in practical problems, for example electromagnetic stirring in a weld pool (Kublanov & Erokhin 1974), the flow takes place in a container which can be approximated by a hemispherical bowl. The current is discharged into the pool from the centre of the free surface and in the symmetric case the streamlines in an axial cross-section form two sets of closed loops, one on either side of the axis. The purpose of this paper is to investigate symmetrical electromagnetic stirring in a hemispherical pool. We are able to produce an analytic solution for the case of slow viscous flow. In the case of the nonlinear problem our equations are of mixed type and we construct a numerical solution.

2. Equations of the problem

We consider a hemispherical bowl of radius a full of incompressible conducting fluid of density ρ and kinematic viscosity ν. The plane boundary of the fluid is a free horizontal surface. At the centre of the plane boundary there is a current source supplying to the fluid region an electric current J_0. The source is usually the end of a wire, perpendicular to the plane of the free surface, from which current is discharged into the fluid. We use spherical polar co-ordinates (r, θ, ϕ) with the origin at the current source and the axis $\theta = 0$ along the axis of the bowl. Thus the fluid occupies the region $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, $r \leqslant a$ and the free surface corresponds to $\theta = \frac{1}{2}\pi$. If we assume that the current density $\mathbf{j}$ is purely radial and ignore the effect of the velocity field on the electromagnetic variables we can show that

$$\mathbf{j} = \hat{\mathbf{r}} J_0 / 2\pi r^2. \tag{1}$$

The associated magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \times 2J_0 (1-\mu)/r(1-\mu^2)^{\frac{1}{2}}, \tag{2}$$

where $\mu = \cos\theta$. The velocity field $\mathbf{v}$ is symmetric about the axis $\theta = 0$ and in terms of a stream function ψ is given by

$$\mathbf{v} = -\left(\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, \frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial r}, 0\right). \tag{3}$$

It was shown by Sozou (1971) that, on dimensional grounds, when the fluid extends to infinity the appropriate form for ψ is $\nu r g_0(\mu)$, where g_0 is a function to be determined. In the present configuration it is convenient to set

$$\psi = \nu r g(\mu, \lambda), \tag{4}$$

where

$$\lambda = r/a. \tag{5}$$

Thus (3) becomes

$$\mathbf{v} = (-\nu/r)\,[g_\mu, (g+\lambda g_\lambda)/(1-\mu^2)^{\frac{1}{2}}, 0], \tag{6}$$

where a suffix λ or μ indicates partial differentiation with respect to that variable. On taking the curl of the steady-state momentum equation

$$\rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B} - \nu\rho\nabla\times\nabla\times\mathbf{v}, \tag{7}$$

and making use of (1), (2), (5) and (6), after a little algebra we obtain

$$(1-\mu^2)f_{\mu\mu} - 4\mu f_\mu + \lambda^2 f_{\lambda\lambda} - 2\lambda f_\lambda - K/(1+\mu) = 3fg_\mu + gf_\mu + \lambda(f_\mu g_\lambda - f_\lambda g_\mu), \tag{8}$$

where p denotes the pressure, $K = 2J_0^2/\pi\rho\nu^2$ and

$$f = g_{\mu\mu} + (2\lambda g_\lambda + \lambda^2 g_{\lambda\lambda})/(1-\mu^2). \tag{9}$$

Equations (8) and (9) are the fundamental equations of our problem. We note that at $r \ll a$, that is, in the limit when $\lambda \to 0, g(\mu, 0) = g_0(\mu)$ and (9) becomes $f = g_0''$, where a prime denotes differentiation. Equation (8) then becomes

$$(1-\mu^2)g_0^{\text{iv}} - 4\mu g_0''' - K/(1+\mu) = 3g_0''g_0 + gg_0'''. \tag{10}$$

When (10) is integrated three times we obtain (Sozou 1971)

$$g_0^2 - 2(1-\mu^2)g_0' - 4\mu g_0 = K[a\mu^2 + b\mu + c - (1+\mu)^2 \log(1+\mu)], \tag{11}$$

where a, b and c are constants of integration.

Equations (8) and (9) must be solved under the following boundary conditions:

$$g(1,\lambda) = 0, \quad g(\mu, 1) = 0, \quad g_\lambda(\mu, 1) = 0, \qquad (12)\text{–}(14)$$

$$g(0,\lambda) = 0, \quad g_{\mu\mu}(0,\lambda) = 0, \qquad (15), (16)$$

$$g(\mu, 0) = g_0(\mu), \quad f(\mu, 0) = g_0'', \qquad (17), (18)$$

$$-4f_\mu + \lambda^2 f_{\lambda\lambda} - 2\lambda f_\lambda - \tfrac{1}{2}K = 3fg_\mu - \lambda f_\lambda g_\mu \quad \text{on} \quad \mu = 1. \tag{19}$$

We note that $g(1,\lambda) = 0$ implies $g_\lambda(1,\lambda) = 0$, $g(0,\lambda) = 0$ implies $g_\lambda(0,\lambda) = 0$ and $g(\mu, 1) = 0$ implies $g_\mu(\mu, 1)$. Equation (12), in conjunction with $g_\lambda(1,\lambda) = 0$, expresses the fact that $\mathbf{v}$ is finite on $\mu = 1$. Equations (13) and (14) imply that $\mathbf{v} = 0$ on the curved surface of the hemisphere ($\lambda = 1$ or $r = a$), whereas (15) means that at the free surface $\mathbf{v}$ is tangential. Equation (16) represents the condition that at the free surface the shear viscous stress is zero. At the free surface we must also have continuity of the normal stress. This is achieved by a suitable deformation of the free surface. Since it is assumed that the free surface is plane, validity of the solution requires that the deformation be small. Thus, for small deformations, continuity of the normal stress is used to obtain the shape of the free surface. Equation (19) is derived from (8), in conjunction with (12), and expresses the fact that $f_{\mu\mu}$ is finite on $\mu = 1$.

In order to evaluate the right-hand sides of (17) and (18) we must determine the constants a, b and c occurring in (11). When (15) and (16) are applied at $\lambda = 0$ we obtain $b = 1$. Since $g_0(1) = 0$, the left-hand side of (11) has a double zero at $\mu = 1$ and thus so must its right-hand side, that is we must have

$$a + b + c - 4\log 2 = 0, \quad 2a + b - 4\log 2 - 2 = 0.$$

Thus $a = \frac{1}{2} + 2\log 2$, $b = 1$ and $c = -\frac{3}{2} + 2\log 2$.

In order to solve (11) we follow Sozou (1971), that is we set

$$g_0 = -2(1-\mu^2)u'/u \tag{20}$$

and thus transform (11) into

$$u'' = \frac{Ku}{4(1-\mu^2)^2}[a\mu^2 + b\mu + c - (1+\mu)^2 \log(1+\mu)]. \tag{21}$$

Equation (21) is solved by forward integration subject to the conditions $u(0) = 1$ and $u'(0) = 0$. The coefficient of u on the right-hand side of (21) is negative for all $0 \leqslant \mu \leqslant 1$ and thus when K is sufficiently large, say $K = K_{\text{crit}}$, $u(1) = 0$, that is $g_0(1) \neq 0$. When $K = K_{\text{crit}}$ we have velocity breakdown near the current source along the axis $\mu = 1$. We find that for the configuration studied here $K_{crit} = 94{\cdot}1$. In view of this breakdown near the current source we cannot solve (8) and (9) for values of K exceeding 94·1. It is quite possible that a value of $K < 94{\cdot}1$ may cause velocity breakdown away from the source and near the solid boundary. This value, if it exists, must be determined from the complete solution of (8) and (9).

3. The linear problem

Equations (8) and (9), under the boundary conditions (12)–(19), are very complex and it is obvious that they must be solved numerically. Fortunately we can construct an exact solution of this problem for the case where the inertia terms in the momentum equation, that is the terms on the right-hand side of (8), are negligible. In this case, of course, the function g_0 occurring on the right-hand sides of (17) and (18) must be constructed from (10) or (11) by neglecting their nonlinear terms. Here it is convenient to work explicitly in terms of g only, instead of f and g, and express the curl of (7), namely

$$\nabla \times \nabla \times \nabla \times \mathbf{v} = \nabla \times (\mathbf{j} \times \mathbf{B})/\nu\rho, \tag{22}$$

as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{(1-\mu^2)}{r^2}\frac{\partial^2}{\partial \mu^2}\right]^2 (rg) = \frac{K}{r^3}(1-\mu). \tag{23}$$

The solution of (23) that satisfies (15) and has the appropriate singularity at the origin is given by

$$2g/K = h(\mu) + (1-\mu^2)\sum_{n=1}^{\infty}[A_{2n}\lambda^{2n} + C_{2n}\lambda^{2n+2}]P'_{2n}(\mu), \tag{24}$$

where

$$h(\mu) = (1+\mu)\log(1+\mu) + A\mu^2 + C\mu, \tag{25}$$

A, C, A_{2n} and C_{2n} are constants to be determined and $P_{2n}(\mu)$ is the Legendre polynomial of degree $2n$.

Equations (12) and (16) give

$$2\log 2 + A + C = 0, \quad 1 + 2A = 0,$$

respectively. Thus

$$A = -\tfrac{1}{2}, \quad C = \tfrac{1}{2} - 2\log 2. \tag{26}$$

In order to be able to satisfy (13) and (14) we must express $h(\mu)$, over the interval $0 \leqslant \mu \leqslant 1$, in the form

$$h(\mu) = (1-\mu^2)\sum_{1}^{\infty} a_{2n}P'_{2n}(\mu). \tag{27}$$

We define

$$h(\mu) = -(1-\mu)\log(1-\mu) - A\mu^2 + C\mu, \quad -1 \leqslant \mu \leqslant 0. \tag{28}$$

On taking account of the orthogonality properties of the functions $(1-\mu^2)P'_m(\mu)$ over the interval $(-1, 1)$ and substituting in (25) and (28) the values of A and C given by (26), we obtain

$$a_{2n} = \frac{(4n+1)}{2n(2n+1)}\int_0^1 h(\mu)P'_{2n}(\mu)\,d\mu = -\frac{(4n+1)P_{2n}(0)}{4n^2(n+1)(2n-1)(2n+1)^2}. \tag{29}$$

Applying the boundary conditions (13) and (14) to (24), with $h(\mu)$ given by (27), we obtain

$$A_{2n} = -(n+1)a_{2n}, \quad C_{2n} = na_{2n}$$

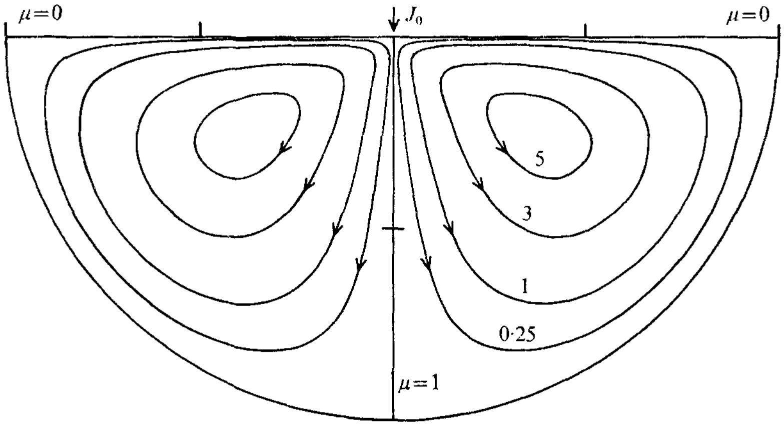

FIGURE 1. Streamlines of the velocity field represented by (30). The numbers on the curves are values of $1000\psi/\nu K$.

and thus

$$a_{2n}+A_{2n}\lambda^{2n}+C_{2n}\lambda^{2n+2} = a_{2n}(1-\lambda^2)^2(1+2\lambda^2+3\lambda^4+\ldots+n\lambda^{2n-2}).$$

Since $P_2(0) = -\frac{1}{2}$ and, for $n \geqslant 2$, $P_{2n}(0) = -(1-(2n)^{-1})P_{2n-2}(0)$, a_{2n} converges very rapidly and we need calculate only the first few terms occurring in the expression for g. If we make use of (27) and terminate (24) at $n = 4$, after a little algebra we obtain

$$\psi/K = \nu r(1-\lambda^2)^2\mu(1-\mu^2)\left[\frac{5}{96}-\frac{3}{2560}(1+2\lambda^2)(7\mu^2-3)\right.$$
$$+\frac{13}{86\,016}(1+2\lambda^2+3\lambda^4)(33\mu^4-30\mu^2+5)-\frac{17}{2\,359\,296}(1+2\lambda^2+3\lambda^4+4\lambda^6)$$
$$\left.\times(715\mu^6-1001\mu^4+385\mu^2-35)\right]. \quad (30)$$

Streamlines of the velocity field represented by (30) are shown in figure 1. As expected the streamlines form two groups of closed loops, one on either side of the axis $\mu = 1$.

The radial component of (7), for the case where the inertia terms are negligible, after a little algebra gives

$$\frac{\partial p}{\partial r} = \frac{\nu^2\rho K}{2r^3}\left[1-2\mu+\sum_{n=1}^{\infty}4n(2n+1)(4n+3)C_{2n}\lambda^{2n+2}P_{2n}(\mu)\right], \quad (31a)$$

or
$$\frac{\partial p}{\partial r} = \frac{2\nu^2\rho K}{r^3}\sum_{n=1}^{\infty}n^2(2n+1)[-2n-1+(4n+3)\lambda^{2n+2}]a_{2n}P_{2n}(\mu). \quad (31b)$$

Hence, apart from an additive constant,

$$p = \frac{\nu^2\rho K}{r^2}\left[-\frac{1-2\mu}{4}+\sum_{n=1}^{\infty}n(2n+1)(4n+3)a_{2n}\lambda^{2n+2}P_{2n}\right], \quad (32a)$$

or
$$p = \frac{\nu^2\rho K}{r^2}\sum_{n=1}^{\infty}n(2n+1)[n(2n+1)+(4n+3)\lambda^{2n+2}]a_{2n}P_{2n}. \quad (32b)$$

Equation ($31a$) is obtained when we use in (24) the form of $h(\mu)$ given by (25) and ($31b$) is obtained when we use the form of $h(\mu)$ given by (27). We can also transform ($31a$) into ($31b$) by expanding $1-2\mu$ over the interval (0, 1) as a series of Legendre polynomials of even degree.

At the free surface the normal hydrodynamic stress

$$p_{\theta\theta} = -p + \frac{2\nu\rho}{r}\left[\mathbf{v}.\hat{\mathbf{r}} + \frac{\partial}{\partial\theta}(\mathbf{v}.\hat{\boldsymbol{\theta}})\right] \tag{33}$$

must be balanced by the surface tension of the suitably deformed surface, that is, at the free surface, we must satisfy the equation

$$p_{\theta\theta} + T(r_1^{-1} + r_2^{-1}) = 0. \tag{34}$$

Here T is the surface tension and r_1 and r_2 are the principal radii of curvature at a general point of the free surface. On $\mu = 0$

$$\mathbf{v}.\hat{\mathbf{r}} + \partial(\mathbf{v}.\hat{\boldsymbol{\theta}})/\partial\theta = 0, \quad r_1^{-1} + r_2^{-1} = 0.$$

Thus on $\mu = 0$,

$$p_{\theta\theta} = \frac{\nu^2\rho K}{r^2}\left[\frac{1}{4} - \sum_{n=1}^{\infty} n(2n+1)(4n+3)\, a_{2n}\lambda^{2n+2}P_{2n}(0)\right]. \tag{35}$$

It can be shown that for $0 \leqslant \lambda \leqslant 1$ the right-hand side of (35) is positive. Thus, for equilibrium, we must apply to the surface $\mu = 0$ an outward normal stress. In the absence of this externally applied stress, the surface $\mu = 0$ will be depressed near $r = 0$, where $p_{\theta\theta}$ is maximum. This is in agreement with experimental observation (Kublanov & Erokhin 1974).

If the depression (or elevation) z of the free surface from the plane $\mu = 0$ satisfies the condition $|z| \ll a$, then z can be calculated as follows. We set

$$z = aH(\varpi), \tag{36}$$

where ϖ is the distance of the point on the free surface from the axis $\mu = \pm 1$. Since $|z| \ll a$, we must have $|H| \ll 1$ and then (34) and (35) give

$$\frac{d}{d\varpi}(\varpi H') = \frac{\nu^2\rho K}{aT}\left[\frac{1}{4}\frac{\varpi}{\varpi^2+H^2} - \sum_{n=1}^{\infty} n(2n+1)(4n+3)\, a_{2n}\varpi(\varpi^2+H^2)^{2n}P_{2n}(0)\right] \tag{37}$$

or

$$\frac{d}{d\varpi}(\varpi H') \simeq \frac{\nu^2\rho K}{aT}\left[\frac{1}{4}\frac{\varpi}{\varpi^2+c^2} - \sum_{n=1}^{\infty} n(2n+1)(4n+3)\, a_{2n}\varpi^{2n+1}P_{2n}(0)\right], \tag{38}$$

where $c = -H(0)$. On integrating (38) twice, subject to the conditions $H'(0) = 0$ and $H(0) = -c$, we obtain

$$H = -c + \frac{\nu^2\rho K}{aT}\left[\frac{1}{8}\int_0^{\varpi}\frac{1}{\varpi}\log\left(1+\frac{\varpi^2}{c^2}\right)d\varpi - \sum_{n=1}^{\infty}\frac{n(2n+1)(4n+3)}{(2n+2)^2}\, a_{2n}\varpi^{(2n+2)}P_{2n}(0)\right]. \tag{39}$$

Since the fluid is incompressible we must have

$$\int_0^1 xH(x)\,dx = 0. \tag{40}$$

c	0·200	0·100	0·050	0·020	0·010	0·005
s	0·707	0·184	0·054	0·013	0·0045	0·0017

TABLE 1. Values of $s = \nu^2\rho K/aT$ for various $c = -H(0)$

After a little manipulation, which also involves reversing the order of integration in a repeated integral, (40) reduces to

$$\frac{8c}{s}+\tfrac{1}{2}(1+c^2)\log(1+c^{-2})-\tfrac{1}{2}+2\sum_{n=1}^{\infty}\frac{n(2n+1)(4n+3)a_{2n}P_{2n}(0)}{(n+1)^2(n+2)} = \int_0^{1/c}\frac{\log(1+t^2)}{t}\,dt, \qquad (41)$$

where $s = \nu^2\rho K/aT = 2J_0^2/\pi\alpha T$. Given s we must solve (41) for c. Then substitution in (39) specifies H. It is, of course, a straightforward matter to solve the inverse problem, that is specify c and obtain s from (41). This we have done and some of our results are shown in table 1.

4. Numerical solution of the nonlinear problem

Within the domain of interest ($0 < \mu < 1$, $0 < \lambda < 1$), for a given g, (8) is an elliptic equation in f and, for a given f, (9) is elliptic in g. We note that g is prescribed on the boundary and that in addition we must satisfy the conditions $g_\lambda(\mu, 1) = 0$ and $g_{\mu\mu}(0,\lambda) = 0$ [(14) and (16)]. It would thus appear that, in the case of (9), g is overspecified on $\lambda = 1$ and on $\mu = 0$. This is, however, compensated for by the fact that there are no explicit conditions for $f(\mu, 1)$ and $f(0,\lambda)$. Thus the conditions $g_\lambda(\mu, 1) = 0$ and $g_{\mu\mu}(0,\lambda) = 0$ can be satisfied by a suitable choice of $f(\mu, 1)$ and $f(0,\lambda)$. For example, if we solve (8) for f under the condition $f(0,\lambda) = 0$ and then, using the values of f, solve (9) for g under the condition $g(0,\lambda) = 0$, the condition $g_{\mu\mu}(0,\lambda) = 0$ will be satisfied automatically.

Equations (8) and (9) become parabolic on the part of the boundary where $\mu = 1$ and on $\lambda = 0$. Equations of a similar form have been tackled numerically by Sozou & Pickering [1975; their equations (6) and (7)]. Here we employ the same numerical techniques, which we briefly outline below.

We set
$$\eta = (1-\mu)^{\frac{1}{2}} \qquad (42)$$
so that (8) becomes

$$(2-\eta^2)F_{\eta\eta}+(2\lambda G_\lambda+2G+6-7\eta^2)F_\eta/\eta+4\lambda^2F_{\lambda\lambda} - 2\lambda(G_\eta/\eta+4)F_\lambda+6G_\eta F/\eta = 4K/(2-\eta^2), \qquad (43)$$

where $F(\eta,\lambda) = f(1-\eta^2,\lambda)$ and $G(\eta,\lambda) = g(1-\eta^2,\lambda)$. Equation (43) is elliptic throughout the region of interest ($0 < \eta < 1$, $0 < \lambda < 1$) and the boundary conditions for the solution of (43) are

$$F(\eta,0) = g_0''(\mu), \qquad (44)$$

$$F(\eta,1) = \begin{cases} g_{\lambda\lambda}(\mu,1)/(1-\mu^2), & \mu \neq 1, \qquad (45a)\\ -\tfrac{1}{2}g_{\mu\lambda\lambda}(\mu,1), & \mu = 1, \qquad (45b)\end{cases}$$

$$F_\eta(0,\lambda) = 0, \quad F(1,\lambda) = 0. \qquad (46), (47)$$

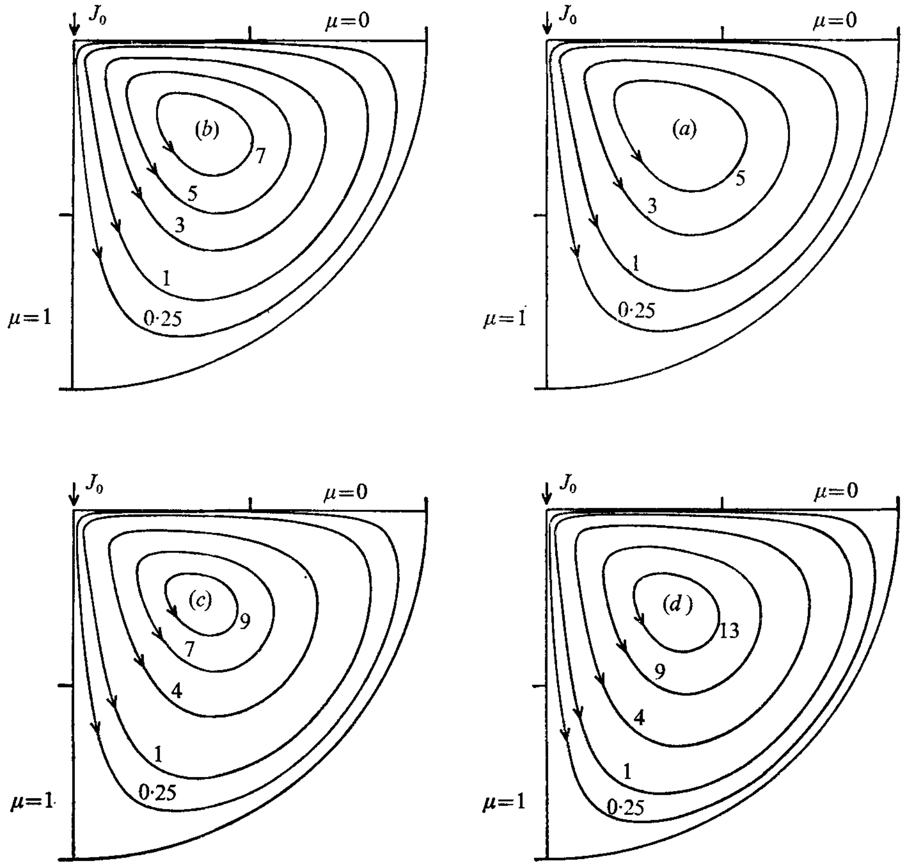

FIGURE 2. Streamlines for the nonlinear problem in half of an axial cross-section. The numbers on the curves are values of $1000\psi/\nu K$. (a) $K = 10$. (b) $K = 15$. (c) $K = 17$. (d) $K = 18{\cdot}5$.

Equation (44) follows immediately from (18), (45a) is derived from (9) using (14), and (45b) is the limiting form of (45a) as $\mu \to 1$. Equation (47) follows from (9) using (15) and (16); (46), together with the requirement that as $\eta \to 0$

$$F_\eta/\eta = F_{\eta\eta}, \quad G_\eta/\eta = G_{\eta\eta}, \qquad (48), (49)$$

replaces (19) and represents the condition that the partial derivatives of f and g with respect to μ are finite on $\mu = 1$.

Equation (9) can also be transformed into the η, λ plane but in view of its relative simplicity and the fact that it is completely elliptic within the region of interest it was tackled in the original μ, λ plane. Equations (14) and (16) were satisfied by assuming

$$g(\mu, 1-\delta\lambda) = \tfrac{1}{2}(\delta\lambda)^2 g_{\lambda\lambda}, \quad g(\delta\mu, \lambda) = (\delta\mu) g_\mu \qquad (50), (51)$$

and using these as boundary conditions; that is we solved (9) in the region $\delta\mu < \mu < 1$, $0 < \lambda < 1-\delta\lambda$, estimating g on $1-\delta\lambda$ and on $\delta\mu$ from (50) and (51), respectively.

λ/K	1	10	15	17	18·5
0	1·009	1·105	1·167	1·195	1·216
0·1	0·994	1·226	1·523	1·792	2·292
0·2	0·929	1·162	1·495	1·843	2·692
0·3	0·824	1·035	1·350	1·704	2·715
0·4	0·690	0·867	1·137	1·454	2·457
0·5	0·539	0·677	0·888	1·145	2·013
0·6	0·383	0·480	0·632	0·818	1·477
0·7	0·236	0·297	0·390	0·507	0·933
0·8	0·114	0·144	0·189	0·246	0·457
0·9	0·030	0·038	0·050	0·065	0·121

TABLE 2. Values of $-10g_\mu/K$ along the axis $\mu = 1$ for the nonlinear problem

Equations (9) and (43) were solved iteratively by successive over-relaxation as follows. We specified an initial approximation to g and used (45) to estimate $F(\eta, 1)$. Equation (43) was then solved for F. This solution was substituted in (9), which was then solved for a better approximation to g. This solution for g was used for a new estimate of $F(\eta, 1)$ from (45) and for constructing an improved solution for F to be used in a new approximation to g and so on. For each iteration the left-hand sides of (50) and (51) were estimated by using the values of $g_{\lambda\lambda}$ and g_μ obtained from the preceding iteration. This process was repeated until convergence, that is until mesh-point values of F and g for two successive iterations changed by less than 1 %. The numerical techniques used for the expression of (9) and (43) as finite-difference equations were very similar to those employed by Sozou & Pickering. The step length in both the λ and η direction was 0·05.

5. Results and discussion

We have computed the flow field for the nonlinear problem for several values of K. Some of our results are shown in figure 2 and table 2. Figure 2 shows streamlines for the cases $K = 10, 15, 17$ and 18·5. Table 2 shows values of g_μ/K along the axis $\mu = 1$ for the cases $K = 1, 10, 15, 17$ and 18·5. If g^l and g^n denote the value of g obtained from the solutions of the linear and nonlinear problems, respectively, and we define $R = g^n/g^l$, we find that for a given λ and K, R attains its greatest value on $\mu = 1$. As $K \to 0$, $R \to 1$ as expected. $\partial R/\partial K > 0$, though when K is of order unity the nonlinear flow field is indistinguishable from the linear one, which is shown in figure 1.

In the linear case

$$\partial g/\partial \lambda = \partial(\psi/vr)/\partial\lambda = 0 \quad \text{at} \quad \lambda = 0,$$

as can easily be seen from (30); that is, for a given direction μ, g is greatest at the origin. In the nonlinear problem, as K increases so does g/K and the value of λ at which it reaches its greatest value in a given direction. For example, when $K = 1$ the maximum value of $|g'_\mu|$ on $\mu = 1$ occurs at $\lambda = 0·05$.

As K increases from unity there is an accelerating increase in the value of $|g_\mu|/K$ and the nonlinearities of the problem become more pronounced, especially at some distance from the origin, as can easily be verified by inspection of table 2. It is particularly noticeable in table 2 that, on $\mu = 1$ for $\lambda > 0{\cdot}4$, as K increases from 15 to 17, that is, by about 13 %, there is an increase in the value of $|g_\mu|/K$ of the order of 30 %, whereas when K increases from 17 to 18·5, that is, by about 9 %, the value of $|g_\mu|/K$ increases by more than 70 %. Since $\mathbf{v}.\hat{\mathbf{r}} = -\nu g_\mu/r$ and $\mathbf{v} = 0$ at $\lambda = 1$, $\mu = 1$, this accelerating increase (with respect to K) in the value of $|g_\mu|$ must soon lead to a very steep velocity gradient at $\lambda = 1$, $\mu = 1$, which will cause velocity breakdown. It would be very difficult to estimate precisely the value of K, say K^c, that gives rise to velocity breakdown at the vertex of the container but we suspect that it satisfies the condition $18{\cdot}5 < K^c < 20$. We must finally draw attention to the fact that not all the discharged current is used to drive the velocity field and thus, in practice, it is quite probable (Sozou 1974) that a current J_0 larger than that given by $K^c = 2J_0^2/\pi\rho\nu^2$ is required to produce a velocity breakdown at $\lambda = 1$, $\mu = 1$.

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